

# Spectra generated by a confined softcore Coulomb potential

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Analytic and approximate solutions for the energy eigenvalues generated by a confined softcore Coulomb potentials of the form  $a/(r + \beta)$  in  $d > 1$  dimensions are constructed. The confinement is effected by linear and harmonic-oscillator potential terms, and also through ‘hard confinement’ by means of an impenetrable spherical box. A byproduct of this work is the construction of polynomial solutions for a number of linear differential equations with polynomial coefficients, along with the necessary and sufficient conditions for the existence of such solutions. Very accurate approximate solutions for the general problem with arbitrary potential parameters are found by use of the asymptotic iteration method.

PACS numbers: 31.15.-p 31.10.+z 36.10.Ee 36.20.Kd 03.65.Ge.

Keywords: softcore Coulomb potentials, confined potentials, biconfluent Heun equation, asymptotic iteration method, polynomial solutions of differential equations.

## I. INTRODUCTION

### A. Confined atoms in $d = 3$ dimensions

In order to fix ideas we considered first a simple model [17] for a soft confined atom obeying a Schrödinger equation in  $d = 3$  dimensions of the form  $(-\Delta + V)\psi = E\psi$ , where  $V(r)$  is an attractive central potential with Coulomb and confining terms. If we assume a wave function of the form  $\psi(r) = Y_\ell^m(\theta, \phi)r^\ell \exp(-g(r))$ , then we find the radial eigenequation implies

$$(E - V(r))r = rg''(r) + 2(l + 1)g'(r) - r(g'(r))^2. \quad (1)$$

If we now choose  $g(r) = \frac{1}{2}(vr + \omega r^2)$ ,  $v > 0$ ,  $\omega > 0$ , we obtain the following family of exact solutions

$$V(r) = v \left( -\frac{\ell + 1}{r} + \omega r \right) + \omega^2 r^2, \quad E = (3 + 2\ell)\omega - v^2/4, \quad l = 0, 1, 2, \dots \quad (2)$$

The lowest radial excitations of the familiar Coulomb and oscillator problems are recovered from the special cases  $\omega = 0$  or  $v = 0$ . Such specific exact solutions allow for analytical reasoning and explorations. In addition, the explicit results provide relevant test problems for the complementary approaches that must be used to complete the solution space. The question of the existence of exact solutions and the methods for finding them are therefore an important part of the overall task. As in the choice of the function  $g(r)$  in the simple illustration above, it is often the case in this context that exactness has something to do with polynomials. Thus part of the paper involves the issue of when an ordinary differential equation admits polynomial solutions. As we shall see, the ‘asymptotic iteration method’ [11] plays a role both in the construction of exact solutions and also in finding approximations for arbitrary values of the problem parameters.

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## B. Formulation of the problem in $d$ dimensions

The Schrödinger equation in  $d > 1$  dimensions, in atomic units  $\hbar = 2\mu = 1$ , with a spherically symmetric potential  $V(r)$  can be written as [13]

$$H\psi \equiv [-\Delta_d + V(r)]\psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad (3)$$

where  $\Delta_d$  is the  $d$ -dimensional Laplacian operator and  $\mathbf{r} = (x_1, x_2, \dots, x_d)$ ,  $r^2 = \|\mathbf{r}\|^2 = \sum_{i=1}^d x_i^2$ . The quantum wave function  $\psi$  is an element of the Hilbert space  $L^2(\mathbb{R}^d)$ . The principal class of spherically symmetric confining potentials we shall consider has the form

$$V(r) = \frac{a}{r + \beta} + cr + b^2 r^2, \quad \beta > 0, \quad b > 0. \quad (4)$$

Thus  $V(r)$  is continuous and  $V(r) \rightarrow \infty$ . Consequently, by Theorem XIII.67 of Reed-Simon-IV [25], we know that  $H$  has purely discrete eigenvalues and a complete set of eigenfunctions. Meanwhile, for  $d \geq 3$ , by Theorem XIII.69 of the same reference, we have a similar conclusion if we admit the Coulomb singularity in  $V(r)$  by allowing  $\beta = 0$ . For the case of hard confinement,  $r \leq R < \infty$  with Dirichlet boundary conditions at  $r = R$ , and  $\beta > 0$ , so that  $V(r)$  is continuous, we know from Theorem 23.56 of Ref.[16] that again  $H$  has a purely discrete spectrum. These general results cover the cases we consider in this paper. A comparable class of potentials has been carefully analysed in Refs.[1, 7]. In order to express (3) in terms of  $d$ -dimensional spherical coordinates  $(r, \theta_1, \theta_2, \dots, \theta_{d-1})$ , we separate variables using

$$\psi(\mathbf{r}) = r^{-(d-1)/2} u(r) Y_{\ell_1, \dots, \ell_{d-1}}(\theta_1 \dots \theta_{d-1}), \quad (5)$$

where  $Y_{\ell_1, \dots, \ell_{d-1}}(\theta_1 \dots \theta_{d-1})$  is a normalized spherical harmonic [4] with characteristic value  $\ell(\ell + d - 2)$ , and  $\ell = \ell_1 = 0, 1, 2, \dots$  (the angular quantum numbers). One obtains the radial Schrödinger equation as

$$\left[ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + V(r) - E \right] u_{n\ell}^{(d)}(r) = 0, \quad \int_0^\infty \left\{ u_{n\ell}^{(d)}(r) \right\}^2 dr = 1, \quad u_{n\ell}^{(d)}(0) = 0, \quad (6)$$

where  $k = d + 2\ell$ . Since the potential  $V(r)$  is less singular than the centrifugal term,

$$u(r) \sim A r^{(k-1)/2}, \quad r \rightarrow 0, \quad \text{where } A \text{ is a constant.}$$

We note that the Hamiltonian and boundary conditions of (6) are invariant under the transformation

$$(d, \ell) \rightarrow (d \mp 2, \ell \pm 1),$$

thus, given any solution for fixed  $d$  and  $\ell$ , we can immediately generate others for different values of  $d$  and  $\ell$ . Further, the energy is unchanged if  $k = d + 2\ell$  and the number of nodes  $n$  is constant. Repeated application of this transformation produces a large collection of states; this has been discussed, for example, in Ref. [14].

In the present work we study the exact and approximate solutions of the Schrödinger eigenproblem generated by a confined soft-core Coulomb potential in  $d$ -dimensions, where  $d > 1$ . As we have discussed above, for the cases we consider, the spectrum of this problem is discrete, all eigenvalues are real and simple, and they can be arranged in an increasing sequence  $\lambda_0 < \lambda_1 < \dots \rightarrow \infty$ . The paper is organized as follows. In section II, we set up the Schrödinger equation for the potential (4) and discuss the correspondence second-order differential equation. In section III, we present our method of solution that relies on the analysis of polynomial solutions of the differential equation

$$\left( \sum_{i=0}^k a_{k,i} r^{k-i} \right) y'' + \left( \sum_{i=0}^{k-1} a_{k-1,i} r^{k-1-i} \right) y' - \left( \sum_{i=0}^{k-2} \tau_{k-2,i} r^{k-2-i} \right) y = 0, \quad k \geq 2 \quad (7)$$

and different variants of this general differential-equation class. We discuss in particular necessary and sufficient conditions on the equation parameters for it to have polynomial solutions. A brief review of the asymptotic iteration method (AIM) is presented in section IV. In section V, the exact and approximate solutions for the problem are discussed, based on the results of section II; and approximate solutions are found for arbitrary potential parameters  $a, b, c$  and  $\beta$  by an application of AIM. An analysis of the corresponding exact and approximate solutions for the pure confined Coulomb case  $\beta = 0$  is presented in section VI. The ‘hard confinement’ case, that is to say when the same system confined to the interior of an impenetrable spherical box of radius  $R$ , is discussed in section VII. In each of these sections, the results obtained are of two types: exact analytic results that are valid when certain parametric constraints are satisfied, and accurate numerical values for arbitrary sets of potential parameters.

## II. SETTING UP THE DIFFERENTIAL EQUATION

In this section, we consider the  $d$ -dimensional radial Schrödinger equation for  $d > 1$  :

$$\left[ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + \frac{a}{r+\beta} + cr + b^2r^2 \right] u_{nl}^d(r) = E_{nl}^d u_{nl}^d(r), \quad \beta > 0, \quad a \in (-\infty, \infty), \quad 0 < r < \infty, \quad u(0) = 0. \quad (8)$$

We note first that the differential equation (8) has one regular singular point at  $r = 0$  with exponents given by the roots of the indicial equation

$$s(s-1) - \frac{1}{4}(k-1)(k-3) = 0, \quad (9)$$

and an irregular singular point at  $r = \infty$ . For large  $r$ , the differential equation (8) assumes the asymptotic form

$$\left[ -\frac{d^2}{dr^2} + cr + b^2r^2 \right] u_{nl}^d(r) \approx 0 \quad (10)$$

with an asymptotic solution

$$u_{nl}^d(r) \approx \exp \left( -\frac{c}{2b}r - \frac{b}{2}r^2 \right). \quad (11)$$

The roots  $s$  of Eq.(9), namely,

$$s_1 = \frac{1}{2}(3-k), \quad s_2 = \frac{1}{2}(k-1).$$

determine the behaviour of  $u_{nl}^d(r)$  as  $r$  approaches 0, only  $s \geq 1/2$  is acceptable, since only in this case is the mean value of the kinetic energy finite [21]. Thus, the exact solution of (8) will assume the form

$$u_{nl}^d(r) = r^{(k-1)/2} \exp \left( -\frac{c}{2b}r - \frac{b}{2}r^2 \right) f_n(r), \quad k = d + 3l, \quad (12)$$

where we note that  $u_{nl}^d(r) \sim r^{(k-1)/2}$  as  $r \rightarrow 0$ . On insertion of this ansatz wave function into (8), we obtain the differential equation for  $f_n(r)$  as

$$\begin{aligned} & -4b^2r(r+\beta)f_n''(r) + (8b^3r^3 + 4b(c+2b^2\beta)r^2 + 4b(b+c\beta-bk)r + 4b^2\beta(1-k))f_n'(r) \\ & + ([4b^2(bk-E)-c^2]r^2 + (4ab^2-\beta c^2-4b^2\beta E + 2bc(k-1) + 4b^3\beta k)r + 2b\beta c(k-1))f_n(r) = 0. \end{aligned} \quad (13)$$

In the next section, we study the polynomial solutions of this differential equation which itself lies within a larger class of differential equations given by

$$(a_{4,2}r^2 + a_{4,3}r)y'' + (a_{3,0}r^3 + a_{3,1}r^2 + a_{3,2}r + a_{3,3})y' - (\tau_{2,0}r^2 + \tau_{2,1}r + \tau_{2,2})y = 0, \quad (14)$$

where  $\tau_{2,0}, \tau_{2,1}, \tau_{2,2}$  and  $a_{i,j}$  are real constants for  $i = 3, 4$  and  $j = 0, 1, 2, 3$ .

## III. THE METHOD OF SOLUTION

The necessary condition ([12], Theorem 6) for polynomial solutions  $y(r) = \sum_{k=0}^n c_k r^k$  of the second-order linear differential equation (14) is

$$\tau_{2,0} = n a_{3,0}, \quad n = 0, 1, 2, \dots, \quad (15)$$

provided  $a_{3,0}^2 + \tau_{2,0}^2 \neq 0$ . The polynomial coefficients  $c_n$  then satisfy the four-term recurrence relations

$$\begin{aligned} & ((n-2)a_{3,0} - \tau_{2,0})c_{n-2} + ((n-1)a_{3,1} - \tau_{2,1})c_{n-1} + (n(n-1)a_{4,2} + na_{3,2} - \tau_{2,2})c_n \\ & + (n(n+1)a_{4,3} + (n+1)a_{3,3})c_{n+1} = 0, \quad c_{-2} = c_{-1} = 0. \end{aligned} \quad (16)$$

The proof of (16) follows from an application of the Frobenius method. We note that the recurrence relations (16) can be written as a system of linear equations in the unknown coefficients  $c_i$ ,  $i = 0, \dots, n$  given by

$$\begin{bmatrix} \gamma_0 & \delta_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \beta_1 & \gamma_1 & \delta_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \beta_n & \gamma_n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (17)$$

where

$$\gamma_n = n(n-1)a_{4,2} + na_{3,2} - \tau_{2,2}, \quad \delta_n = n(n+1)a_{4,3} + (n+1)a_{3,3}, \quad \beta_n = (n-1)a_{3,1} - \tau_{2,1}, \quad \alpha_n = (n-2)a_{3,0} - \tau_{2,0}. \quad (18)$$

Thus, for zero-degree polynomials  $c_0 \neq 0$  and  $c_n = 0$ ,  $n \geq 1$ , we must have  $\gamma_0 = \beta_1 = \alpha_2 = 0$ , thus, in addition to the necessary condition  $\tau_{2,0} = 0$ , the following two conditions become sufficient

$$\tau_{2,2} = 0, \quad \tau_{2,1} = 0. \quad (19)$$

For the first-degree polynomial solution,  $c_1 \neq 0$ , and  $c_n = 0$ ,  $n \geq 2$ , we must have  $\gamma_0 c_0 + \delta_0 c_1 = 0$ ,  $\beta_1 c_0 + \gamma_1 c_1 = 0$ ,  $\alpha_2 c_0 + \beta_2 c_1 = 0$  and  $\alpha_3 c_1 = 0$ , thus, in addition to the necessary condition  $\alpha_3 = 0$  or

$$\tau_{2,0} = a_{3,0}, \quad (20)$$

it is also required that the following two  $2 \times 2$ -determinants simultaneously vanish

$$\begin{vmatrix} -\tau_{2,2} & a_{3,3} \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} -\tau_{2,2} & a_{3,3} \\ -a_{3,0} & a_{3,1} - \tau_{2,1} \end{vmatrix} = 0. \quad (21)$$

For the second-degree polynomial solution,  $c_2 \neq 0$  and  $c_n = 0$  for  $n \geq 3$ , it is necessary that  $\gamma_0 c_0 + \delta_0 c_1 + \mu_0 c_2 = 0$ ,  $\beta_1 c_0 + \gamma_1 c_1 + \delta_1 c_2 = 0$ ,  $\alpha_2 c_0 + \beta_2 c_1 + \gamma_2 c_2 = 0$ ,  $\alpha_3 c_1 + \beta_3 c_2 = 0$ , and  $\alpha_4 c_3 = 0$ , from which we have the necessary condition

$$\tau_{2,0} = 2a_{3,0} \quad (22)$$

along with the vanishing of the two  $3 \times 3$ -determinants

$$\begin{vmatrix} -\tau_{2,2} & a_{3,3} & 0 \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} & 2a_{4,3} + 2a_{3,3} \\ -2a_{3,0} & a_{3,1} - \tau_{2,1} & 2a_{4,2} + 2a_{3,2} - \tau_{2,2} \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} -\tau_{2,2} & a_{3,3} & 2a_{4,4} \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} & 2a_{4,3} + 2a_{3,3} \\ 0 & -a_{3,0} & 2a_{3,1} - \tau_{2,1} \end{vmatrix} = 0, \quad (23)$$

For the third-degree polynomial solution,  $c_3 \neq 0$  and  $c_n = 0$  for  $n \geq 4$ , we then have the necessary condition

$$\tau_{2,0} = 3a_{3,0} \quad (24)$$

along with the vanishing of the two  $4 \times 4$ -determinants,

$$\begin{vmatrix} -\tau_{2,2} & a_{3,3} & 0 & 0 \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} & 2a_{4,3} + 2a_{3,3} & 0 \\ -3a_{3,0} & a_{3,1} - \tau_{2,1} & 2a_{4,2} + 2a_{3,2} - \tau_{2,2} & 3a_{3,3} + 6a_{4,3} \\ 0 & -2a_{3,0} & 2a_{3,1} - \tau_{2,1} & 3a_{3,2} + 6a_{4,2} - \tau_{2,2} \end{vmatrix} = 0 \quad (25)$$

and

$$\begin{vmatrix} -\tau_{2,2} & a_{3,3} & 0 & 0 \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} & 2a_{4,3} + 2a_{3,3} & 0 \\ -3a_{3,0} & a_{3,1} - \tau_{2,1} & 2a_{4,2} + 2a_{3,2} - \tau_{2,2} & 3a_{3,3} + 6a_{4,3} \\ 0 & 0 & -a_{3,0} & 3a_{3,1} - \tau_{2,1} \end{vmatrix} = 0 \quad (26)$$

For the fourth-degree polynomial solution ( $n = 4$ ),  $c_4 \neq 0$  and  $c_n = 0$  for  $n \geq 5$ , we then have the necessary condition

$$\tau_{2,0} = 4a_{3,0} \quad (27)$$

along with the vanishing of the two  $5 \times 5$ -determinants,

$$\begin{vmatrix} -\tau_{2,2} & a_{3,3} & 0 & 0 & 0 \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} & 2a_{4,3} + 2a_{3,3} & 0 & 0 \\ -4a_{3,0} & a_{3,1} - \tau_{2,1} & 2a_{4,2} + 2a_{3,2} - \tau_{2,2} & 3a_{3,3} + 6a_{4,3} & 0 \\ 0 & -3a_{3,0} & 2a_{3,1} - \tau_{2,1} & 3a_{3,2} + 6a_{4,2} - \tau_{2,2} & 4a_{3,3} + 12a_{4,3} \\ 0 & 0 & -2a_{3,0} & 3a_{3,1} - \tau_{2,1} & 4a_{3,2} + 12a_{4,2} - \tau_{2,2} \end{vmatrix} = 0 \quad (28)$$

and

$$\begin{vmatrix} -\tau_{2,2} & a_{3,3} & 2a_{4,4} & 0 & 0 \\ -\tau_{2,1} & a_{3,2} - \tau_{2,2} & 2a_{4,3} + 2a_{3,3} & 0 & 0 \\ -4a_{3,0} & a_{3,1} - \tau_{2,1} & 2a_{4,2} + 2a_{3,2} - \tau_{2,2} & 3a_{3,3} + 6a_{4,3} & 0 \\ 0 & -3a_{3,0} & 2a_{3,1} - \tau_{2,1} & 3a_{3,2} + 6a_{4,2} - \tau_{2,2} & 4a_{3,3} + 12a_{4,3} \\ 0 & 0 & 0 & -a_{3,0} & 4a_{3,1} - \tau_{2,1} \end{vmatrix} = 0 \quad (29)$$

Similar expressions for higher-order polynomial solutions can be easily obtained. The vanishing of these determinants can be regarded as the sufficient conditions under which the coefficients  $\tau_{2,1}$  and  $\tau_{2,2}$  of Eq. (13) can be expressed in terms of the other parameters.

#### IV. THE ASYMPTOTIC ITERATION METHOD AND SOME RELATED RESULTS

The asymptotic iteration method (AIM) is an iterative algorithm originally introduced [11] to investigate the analytic and approximate solutions of the differential equation

$$y'' = \lambda_0(r)y' + s_0(r)y, \quad (' = \frac{d}{dr}) \quad (30)$$

where  $\lambda_0(r)$  and  $s_0(r)$  are  $C^\infty$ -differentiable functions. A key feature of this method is to note the invariant structure of the right-hand side of (30) under further differentiation. Indeed, if we differentiate (30) with respect to  $r$ , we obtain

$$y''' = \lambda_1(r)y' + s_1(r)y \quad (31)$$

where  $\lambda_1 = \lambda_0' + s_0 + \lambda_0^2$  and  $s_1 = s_0' + s_0\lambda_0$ . Further differentiation of equation (31), we obtain

$$y^{(4)} = \lambda_2(r)y' + s_2(r)y \quad (32)$$

where  $\lambda_2 = \lambda_1' + s_1 + \lambda_0\lambda_1$  and  $s_2 = s_1' + s_0\lambda_1$ . Thus, for  $(n+1)^{th}$  and  $(n+2)^{th}$  derivative of (30),  $n = 1, 2, \dots$ , we have

$$y^{(n+1)} = \lambda_{n-1}(r)y' + s_{n-1}(r)y \quad (33)$$

and

$$y^{(n+2)} = \lambda_n(r)y' + s_n(r)y \quad (34)$$

respectively, where

$$\lambda_n = \lambda_{n-1}' + s_{n-1} + \lambda_0\lambda_{n-1} \quad \text{and} \quad s_n = s_{n-1}' + s_0\lambda_{n-1}. \quad (35)$$

From (33) and (34), we have

$$\lambda_n y^{(n+1)} - \lambda_{n-1} y^{(n+2)} = \delta_n y \quad \text{where} \quad \delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n. \quad (36)$$

Clearly, from (36) if  $y$ , the solution of (30), is a polynomial of degree  $n$ , then  $\delta_n \equiv 0$ . Further, if  $\delta_n = 0$ , then  $\delta_{n'} = 0$  for all  $n' \geq n$ . In an earlier paper [11], we proved the principal theorem of AIM, namely

**Theorem IV.1.** *Given  $\lambda_0$  and  $s_0$  in  $C^\infty(a, b)$ , the differential equation (30) has the general solution*

$$y(r) = \exp\left(-\int^r \frac{s_{n-1}(t)}{\lambda_{n-1}(t)} dt\right) \left[ C_2 + C_1 \int^r \exp\left(\int^t (\lambda_0(\tau) + 2\frac{s_{n-1}(\tau)}{\lambda_{n-1}(\tau)}) d\tau\right) dt \right] \quad (37)$$

if for some  $n > 0$

$$\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0. \quad (38)$$

where  $\lambda_n$  and  $s_n$  are given by (35).

Recently, it has been shown [30] that the termination condition (38) is necessary and sufficient for the differential equation (30) to have polynomial-type solutions of degree at most  $n$ , as we may conclude from Eq.(36). The application of AIM to a number of problems has been outlined in many publications. The applicability of the method is not restricted to a particular class of differentiable functions (e.g. polynomials or rational functions), rather, it can accommodate any type of differentiable function. The fast convergence of the iterative scheme depend on a suitable choice for the starting values of  $r = r_0$  and the correct asymptotic solutions near the boundaries [10].

## V. EXACT AND APPROXIMATE SOLUTIONS FOR THE SOFT-CONFINED SOFTCORE COULOMB POTENTIAL

Comparing equation (13) with (14) and using parameters given by

$$\begin{aligned} a_{4,2} &= -4b^2, & a_{4,3} &= -4b^2\beta, \\ a_{3,0} &= 8b^3, & a_{3,1} &= 4b(c + 2b^2\beta), & a_{3,2} &= 4b(b + c\beta - bk), & a_{3,3} &= 4b^2\beta(1 - k), \\ \tau_{2,0} &= c^2 + 4b^2(E_{nl}^d - bk), & \tau_{2,1} &= -4ab^2 + \beta c^2 + 4b^2\beta E_{nl}^d - 2bc(k - 1) - 4b^3\beta k, & \tau_{2,2} &= -2b\beta c(k - 1), \end{aligned} \quad (39)$$

the exact solution of (8) assumes the following form

$$u_{nl}^d(r) = r^{(k-1)/2} \exp\left(-\frac{c}{2b}r - \frac{b}{2}r^2\right) \sum_{i=0}^{n'} C_i r^i, \quad k = d + 3l, \quad (40)$$

where  $f_{n'}(r) = \sum_{i=0}^{n'} C_i r^i$  and  $n$  counts the number of zeros of  $f_{n'}(r) = 0$ , hence the number of nodes in the wave function solution. The coefficients  $C_i$  can be easily evaluated using the four-term recurrence relations (16),

$$\begin{aligned} 4b^2(i - 2 - n')C_{i-2} + (2ab + c(k - 3 + 2i) + 4b^2\beta(i - 1 - n'))C_{i-1} + (\beta c(k - 1 + 2i) - 2bi(k - 2 + i))C_i \\ - 2b\beta(i + 1)(k - 1 + i)C_{i+1} = 0, \quad C_{-1} = 0, \quad C_0 = 1, \quad C_1 = c/(2b), \quad i \geq 2, \end{aligned} \quad (41)$$

using the necessary condition

$$E_{nl}^d = b(2n' + k) - \frac{c^2}{4b^2}, \quad n' = 0, 1, 2, \dots \quad (42)$$

The potential parameters  $a, b, c$  and  $\beta$  satisfy sufficient conditions according to the following scenarios: for a zero-degree polynomial solution,  $n' = 0$ , if  $f_0(r) = 1$ , the ground-state solution of equation (8) is given by

$$u_{0\ell}^d(r) = r^{(k-1)/2} \exp\left(-\frac{c}{2b}r - \frac{b}{2}r^2\right), \quad (43)$$

with ground-state eigenenergy

$$E_{0\ell}^d = bk - \frac{c^2}{4b^2} \quad \text{subject to the parameter conditions} \quad c = \frac{2ab}{1 - k} \quad \text{and} \quad \beta = 0. \quad (44)$$

In the next section, we shall focus on the case of  $\beta = 0$  which corresponds to the Coulomb potential perturbed by an added polynomial in  $r$ . In the rest of this section, we shall assume  $\beta > 0$ . For a first-degree polynomial solution,  $n' = 1$ ,

$$f_0(r) = 1 + \frac{c}{2b}r, \quad (45)$$

and the exact solution wave function of equation (8) reads

$$u_{0\ell}^d(r) = r^{(k-1)/2} \left(1 + \frac{c}{2b}r\right) \exp\left(-\frac{c}{2b}r - \frac{b}{2}r^2\right), \quad E_{0\ell}^d = b(k + 2) - \frac{c^2}{4b^2}, \quad (46)$$

subject to the following two conditions related the potential parameters

$$4ab^2 + \beta(c^2(k + 1) - 8b^3) = 0 \quad \text{and} \quad 2abc + c^2(k + 1) - 8b^3 = 0. \quad (47)$$

Since, by assumption  $b > 0$  and  $\beta > 0$ ,  $c = 2b/\beta > 0$  and the polynomial solution  $f_i(r) = f_0(r)$  has no roots, in which case  $E_{0\ell}^d$  represent a ground-state solution of the Schrödinger equation (8) subject to the parameters  $a$ ,  $b$  and  $c$  satisfying the conditions given by (47). In summary, the exact solutions of Schrödinger's equation

$$\left[ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + \frac{2b\beta^2 - (k+1)}{\beta(r+\beta)} + \frac{2b}{\beta}r + b^2r^2 \right] u_{0\ell}^d(r) = E_{0\ell}^d u_{0\ell}^d(r), \quad \beta > 0, \quad 0 < r < \infty. \quad (48)$$

is explicitly given by

$$u_{0\ell}^d(r) = r^{(k-1)/2} \left( 1 + \frac{r}{\beta} \right) \exp \left( -\frac{r}{\beta} - \frac{b}{2} r^2 \right), \quad E_{0\ell}^d = b(k+2) - \frac{1}{\beta^2}, \quad (49)$$

In Figure 1, we display the un-normalized ground-state solution using (49) for  $b = \beta = 1$  and different values of  $k$ . For the rest of the spectrum we use the asymptotic iteration method as described in section IV, starting with

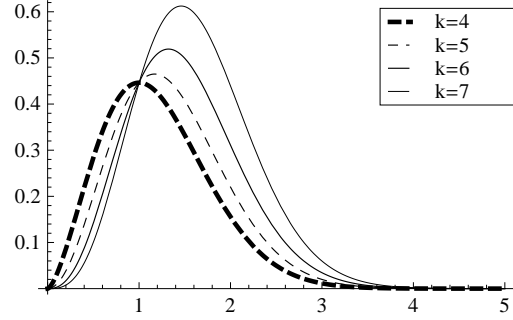


FIG. 1: Un-normalized ground state wave functions as given by (46) for specific values of  $b = \beta = 1$  and different values of  $k$ .

$$u_{nl}^d(r) = r^{(k-1)/2} \left( 1 + \frac{c}{2b} r \right) \exp \left( -\frac{c}{2b} r - \frac{b}{2} r^2 \right) g(r) \quad (50)$$

where  $g(r) = 1$  corresponds to the exact solution (46), equation (48) yields the second-order differential for  $g(r)$  as

$$g''(r) = \left( \frac{2}{\beta} + \frac{1-k}{r} + 2br - \frac{2}{\beta+r} \right) g'(r) + \left( 2b - E + bk - \frac{1}{\beta^2} \right) g(r). \quad (51)$$

Hence, we may initiate AIM with

$$\lambda_0(r) = \frac{2}{\beta} + \frac{1-k}{r} + 2br - \frac{2}{\beta+r} \quad \text{and} \quad s_0(r) = 2b - E + bk - \frac{1}{\beta^2} \quad (52)$$

The question is then to find the initial value  $r_0$  that stabilizes the computation of the termination-condition roots (38). To this end, we take the highest of the absolute values among all the roots of

$$V(r) - E_{0\ell}^d = \frac{(k-1)(k-3)}{4r^2} + \frac{2b\beta^2 - (k+1)}{\beta(r+\beta)} + \frac{2b}{\beta}r + b^2r^2 - \left( b(k+2) - \frac{1}{\beta^2} \right) = 0$$

which yields  $r_0 \sim 3$ , henceforth we shall fix  $r_0$  at  $r_0 = 3$  for all of our numerical computations. In Table I, we report our results from AIM for first 12 decimal places. The eigenvalue reported in Table I were computed using Maple version 16 running on an IBM architecture personal computer and we have chosen a high-precision environment. In order to accelerate our computation we have written our own code for a root-finding algorithm instead of using the default procedure `Solve` of *Maple 16*. The results of AIM may be obtained to any degree of precision, although we have reported our results to only the first twelve decimal places,

For a second-degree polynomial solution,  $n' = 2$ , of (13), we have

$$f_i(r) = 1 + \frac{c}{2b} r + \frac{4ab^2 + \beta((k+1)c^2 - 16b^3)}{8b^2\beta k} r^2, \quad (53)$$

TABLE I: The first few Eigenenergies  $E_{n0}^{d=4}$  of Schrödinger equation (48). The initial value utilize AIM is  $r_0 = 3$ . The subscript  $N$  refers to the number of iteration used by AIM.

$n$	$E_{n0}^{d=4}$	$n$	$E_{n0}^{d=5}$
0	5.000 000 000 000 <sub><math>N=3, Exact</math></sub>	0	6.000 000 000 000 <sub><math>N=3, Exact</math></sub>
1	10.223 655 148 231 <sub><math>N=90</math></sub>	1	11.139 009 555 512 <sub><math>N=81</math></sub>
2	15.140 755 138 866 <sub><math>N=91</math></sub>	2	16.025 939 658 710 <sub><math>N=83</math></sub>
3	19.899 975 543 589 <sub><math>N=92</math></sub>	3	20.771 696 017 356 <sub><math>N=85</math></sub>
4	24.559 997 330 221 <sub><math>N=92</math></sub>	4	25.425 173 414 020 <sub><math>N=110</math></sub>
5	29.150 691 578 737 <sub><math>N=118</math></sub>	4	30.012 690 909 013 <sub><math>N=109</math></sub>

and the exact solution of equation (8) reads

$$u_{i\ell}^d(r) = r^{(k-1)/2} \left( 1 + \frac{c}{2b} r + \frac{4ab^2 + \beta((k+1)c^2 - 16b^3)}{8b^2\beta k} r^2 \right) \exp \left( -\frac{c}{2b} r - \frac{b}{2} r^2 \right), \quad E_{i\ell}^d = b(k+4) - \frac{c^2}{4b^2}, \quad (54)$$

where  $i$  counts the number of roots of the polynomial solution (53) subject to the simultaneous conditions relating the parameters  $a$ ,  $b$ ,  $c$  and  $\beta$ ,

$$\begin{aligned} 4ab^2(-4bk + 3\beta c(1+k)) + \beta^2 c(c^2(1+k)(3+k) - 16b^3(3+2k)) &= 0, \\ 8a^2b^3 + 2ab(-16b^3\beta + \beta c^2(1+k) + 2bc(3+k)) + \beta c(c^2(1+k)(3+k) - 16b^3(3+2k)) &= 0. \end{aligned} \quad (55)$$

In particular, for

$$a = 4b\beta - \frac{\beta c^2}{4b^2} + \left( \frac{c}{b} - \frac{2}{\beta} - \frac{\beta c^2}{4b^2} \right) k$$

The exact solution of the Schrödinger equation for  $0 < r < \infty$

$$\left[ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + \frac{4b\beta - \frac{\beta c^2}{4b^2} + \left( \frac{c}{b} - \frac{2}{\beta} - \frac{\beta c^2}{4b^2} \right) k}{r + \beta} + cr + b^2 r^2 \right] u_{i\ell}^d(r) = \left( b(k+4) - \frac{c^2}{4b^2} \right) u_{i\ell}^d(r) \quad (56)$$

is

$$u_{i\ell}^d(r) = r^{(k-1)/2} \left( 1 + \frac{c}{2b} r + \frac{\beta c - 2b}{2b\beta^2} r^2 \right) \exp \left( -\frac{c}{2b} r - \frac{b}{2} r^2 \right) \quad (57)$$

subject to the relation among the parameters  $b$ ,  $c$  and  $\beta$  given by

$$-16b^3k + 4b^2\beta c(3+5k) + b\beta^2(32b^3 - 8c^2(1+k)) + c\beta^3(c^2(1+k) - 8b^3) = 0 \quad (58)$$

As an example, for  $b = c = 1$  and  $k$ , the roots of equation (58) for  $\beta > 0$  are  $\beta_1 = 0.760\,237\,519\,523\,249\,5$  and  $\beta_2 = 3.854\,071\,917\,077\,363\,6$ . We display in Figure 2 the exact solutions as given by (57). For a third-degree polynomial solution,  $n' = 3$ , of equation (13),

$$f_i(r) = 1 + \frac{c}{2b} r + \frac{4ab^2 + \beta((k+1)c^2 - 24b^3)}{8b^2\beta k} r^2 + \frac{4ab^2(3\beta c(1+k) - 4bk) + \beta^2 c(c^2(1+k)(3+k) - 8b^3(9+7k))}{48b^3\beta^2 k(1+k)} r^3, \quad (59)$$

and the exact solution of equation (8) reads

$$\begin{aligned} u_{i\ell}^d(r) &= r^{(k-1)/2} \exp \left( -\frac{c}{2b} r - \frac{b}{2} r^2 \right) \\ &\quad \left( 1 + \frac{c}{2b} r + \frac{4ab^2 + \beta((k+1)c^2 - 24b^3)}{8b^2\beta k} r^2 + \frac{4ab^2(3\beta c(1+k) - 4bk) + \beta^2 c(c^2(1+k)(3+k) - 8b^3(9+7k))}{48b^3\beta^2 k(1+k)} r^3 \right), \end{aligned} \quad (60)$$



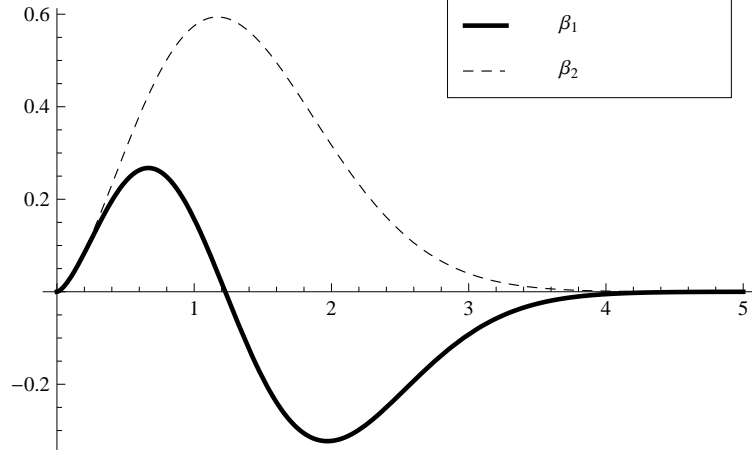


FIG. 2: Un-normalized ground-state and first-excited wave functions as given by (57) for  $k = 4, b = c = 1$  and specific values of  $\beta$ .

where

$$E_{il}^d = b(k+6) - \frac{c^2}{4b^2}, \quad (61)$$

and the parameters  $a, b, c$  and  $\beta$  satisfy, by means of (28) and (29), the conditions

$$\begin{aligned} & 48a^2b^4\beta(1+k) - 8ab^2(48b^3\beta^2(1+k) - 12b^2k(1+k) + 8b\beta ck(2+k) - 3\beta^2c^2(1+k)(3+k)) \\ & + \beta^3(576b^6(1+k) + c^4(1+k)(3+k)(5+k) - 16b^3c^2(24+5k(5+k))) = 0 \\ & 8a^2b^3(-4bk + 3\beta c(1+k)) + 2ab([\beta^2c^3(3+k) - 48b^4\beta](1+k) - 8b^2ck(5+k) + 6b\beta c^2(1+k)(5+k) \\ & - 8b^3\beta^2c(9+7k)) + \beta^2([576b^6 + c^4(3+k)(5+k)](1+k) - 16b^3c^2(24+5k(5+k))) = 0. \end{aligned} \quad (62)$$

For arbitrary values of the potential parameters  $a, b, c$  and  $\beta$  that do not necessarily obey the above conditions, we may use AIM directly to compute the eigenvalues *accurately*, as the zeros of the termination condition (38). The method above can also be used to verify the exact solutions we have obtained earlier. For arbitrary parameters, we employ AIM with

$$\lambda_0(r) = \frac{1-k}{r} + 2br + \frac{c}{b} \quad \text{and} \quad s_0(r) = \frac{c(k-1)}{2br} + \frac{a}{r+\beta} + \frac{4b^3k - c^2 - 4b^2E_{nl}^d}{4b^2}. \quad (63)$$

and compute the AIM sequences  $\lambda_n$  and  $s_n$  as given by Eq.(35). We note that for given values of the potential parameters  $a, b$ , and of  $k = d + 2\ell$ , the termination condition  $\delta_n = \lambda_n s_{n-1} - \lambda_{n-1} s_n = 0$  yields again an expression that depends on both  $r$  and  $E$ . Thus, in order to use AIM as an approximation technique for computing the eigenvalues  $E$  we need to feed AIM with a suitable initial value of  $r = r_0$  that could stabilize AIM (that is, to avoid oscillations). Again, for our calculations in Table II, we have used  $r_0 = 3$ .

## VI. EXACT AND APPROXIMATE SOLUTIONS FOR THE PURE COULOMB POTENTIAL PLUS LINEAR AND OSCILLATOR RADIAL TERMS

In this section, we focus our attention on the case of  $\beta = 0$ , specifically we study the exact and approximate eigenenergies of a hydrogenic atom with a Coulomb potential [9] in the presence of an external linear term and an

TABLE II: Eigenvalues  $E_{0\ell}^{d=4,5}$  for  $V(r) = 1/(r+1) + r + r^2$  and different values of the angular momentum  $\ell$ . The initial value used by AIM is  $r_0 = 3$ . The subscript  $N$  refers to the number of iteration used by AIM.

$\ell$	$E_{0\ell}^{d=4}$	$l$	$E_{0l}^{d=5}$
0	5.743 064 598 822 $_{N=75} = E_{01}^2$	0	6.881 699 763 857 $_{N=69} = E_{01}^3$
1	8.010 441 473 733 $_{N=63} = E_{02}^2 = E_{00}^6$	1	9.131 165 616 720 $_{N=56} = E_{02}^3 = E_{00}^7$
2	10.245 221 261 283 $_{N=52} = E_{03}^2 = E_{01}^6 = E_{00}^8$	2	11.353 616 525 901 $_{N=48} = E_{03}^3 = E_{01}^7 = E_{00}^9$
3	12.457 128 050 974 $_{N=44} = E_{04}^2 = E_{02}^6 = E_{01}^8 = E_{00}^{10}$	3	13.556 369 149 873 $_{N=41} = E_{04}^3 = E_{02}^7 = E_{01}^9 = E_{00}^{11}$
4	14.651 834 191 239 $_{N=39} = E_{05}^2 = E_{03}^6 = E_{02}^8 = E_{01}^{10} = E_{00}^{12}$	4	15.743 928 601 250 $_{N=38} = E_{05}^3 = E_{03}^7 = E_{02}^9 = E_{01}^{11} = E_{00}^{13}$
5	16.832 989 791 994 $_{N=35} = E_{06}^2 = E_{04}^6 = E_{03}^8 = E_{02}^{10} = E_{01}^{12} = E_{00}^{14}$	5	17.919 302 156 447 $_{N=32} = E_{06}^3 = E_{04}^7 = E_{03}^9 = E_{02}^{11} = E_{01}^{13} = E_{00}^{15}$

harmonic oscillator. We have

$$V(r) = \frac{a}{r} + cr + b^2 r^2, \quad a \neq 0, \quad b > 0. \quad (64)$$

This soft-confined potential has been the subject of intensive study over the past few decades in a wide range of contexts [6, 18, 19, 26, 27]. In the light of solutions to the equation (14), we discuss the quasi-exact solutions of Schrödinger equation for the potential (64) and their connection with the solution of the biconfluent heun equation [3, 8, 15, 22–24, 28] where we extend the some of the known results to arbitrary dimensions and provide a compact analytic solutions that we use to verify our approximation method using AIM. For this purpose, we set  $\beta = 0$  in the differential equation (13) to obtain

$$r f_n''(r) + \left(-2b r^2 - \frac{c}{b} r + k - 1\right) f_n'(r) + \left(\left(E - b k + \frac{c^2}{4b^2}\right) r - a + \frac{(1-k)c}{2b}\right) f_n(r) = 0. \quad (65)$$

This equation can easily be compared with (14) for  $a_{4,3} = a_{3,3} = \tau_{3,3} = 0$ , in which case equation (14) reduces to

$$a_{4,2} r y'' + (a_{3,0} r^2 + a_{3,1} r + a_{3,2}) y' + (-\tau_{2,0} r - \tau_{2,1}) y = 0, \quad (66)$$

with polynomial solutions  $y_n = \sum_{j=0}^n C_j r^j$  only if  $\tau_{2,0} = n a_{3,0}$ , and polynomial coefficients  $C_j$  that satisfy the three-term recurrence relations, derived by use of the Frobenius method, given by

$$((n-1)a_{3,0} - \tau_{2,0})C_{n-1} + (na_{3,1} - \tau_{2,1})C_n + (n+1)(na_{4,2} + a_{3,2})C_{n+1} = 0, \quad C_{-1} = 0, \quad C_0 = 1. \quad (67)$$

In this case, the first few polynomials are

$$\begin{aligned}
f_0(r) &= 1 \quad \text{providing} \quad \tau_{2,1} = 0, \\
f_1(r) &= 1 + \frac{\tau_{2,1}}{a_{3,2}} r \quad \text{providing} \quad \begin{vmatrix} -\tau_{2,1} & a_{3,2} \\ -a_{3,0} & a_{3,1} - \tau_{2,1} \end{vmatrix} = 0, \\
f_2(r) &= 1 + \frac{\tau_{2,1}}{a_{3,2}} r + \frac{2a_{3,0}a_{3,2} - a_{3,1}\tau_{2,1} + \tau_{2,1}^2}{2a_{3,2}(a_{3,2} + a_{4,2})} r^2 \quad \text{providing} \quad \begin{vmatrix} -\tau_{2,1} & a_{3,2} & 0 \\ -2a_{3,0} & a_{3,1} - \tau_{2,1} & 2(a_{3,2} + a_{4,2}) \\ 0 & -a_{3,0} & 2a_{3,1} - \tau_{2,1} \end{vmatrix} = 0, \\
f_3(r) &= 1 + \frac{\tau_{2,1}}{a_{3,2}} r + \frac{3a_{3,0}a_{3,2} - a_{3,1}\tau_{2,1} + \tau_{2,1}^2}{2a_{3,2}(a_{3,2} + a_{4,2})} r^2 + \frac{a_{3,0}(-6a_{3,1}a_{3,2} + 7a_{3,2}\tau_{2,1} + 4a_{4,2}\tau_{2,1}) + \tau_{2,1}(2a_{3,1}^2 - 3a_{3,1}\tau_{2,1} + \tau_{2,1}^2)}{6a_{3,2}(a_{3,2} + a_{4,2})(a_{3,2} + 2a_{4,2})} r^3, \\
&\quad \text{providing} \quad \begin{vmatrix} -\tau_{2,1} & a_{3,2} & 0 & 0 \\ -3a_{3,0} & a_{3,1} - \tau_{2,1} & 2a_{3,2} + 2a_{4,2} & 0 \\ 0 & -2a_{3,0} & 2a_{3,1} - \tau_{2,1} & 3a_{3,2} + 6a_{4,2} \\ 0 & 0 & -a_{3,0} & 3a_{3,1} - \tau_{2,1} \end{vmatrix} = 0.
\end{aligned}$$

On other hand, as noted earlier, (65) is a special case of the biconfluent Heun differential equation [20, 28, 29]

$$z f''(z) + (1 + \alpha - \beta z - 2z^2) f'(z) + \left[ (\gamma - \alpha - 2) z - \frac{1}{2}(\delta + (1 + \alpha)\beta) \right] f(z) = 0. \quad (68)$$

Indeed, by a simple comparison, with  $z = \sqrt{b}r$ , between (68) and (64), we find by using

$$\alpha = k - 2, \quad \beta = \frac{c}{b^{3/2}}, \quad \gamma = \frac{E}{b} + \frac{c}{4b^3}, \quad \delta = \frac{2a}{\sqrt{b}}. \quad (69)$$

that we can express the analytic solutions of (66) in terms of the Bi-confluent Heun functions [20, 28, 29] as

$$f(r) = H_B \left( k - 2, \frac{c}{b^{3/2}}, \frac{E}{b} + \frac{c}{4b^3}, \frac{2a}{\sqrt{b}}, \sqrt{b}r \right) \quad (70)$$

with polynomial solutions providing  $E_{nl}^d = b(2n' + k) - c^2/4b^2$ ,  $n' = 0, 1, 2, \dots$ . To this end, the polynomial solutions of the differential equation

$$r f_{n'}''(r) + \left( -2br^2 - \frac{c}{b}r + k - 1 \right) f_{n'}'(r) + \left( 2bn'r - a + \frac{(1-k)c}{2b} \right) f_{n'}(r) = 0, \quad n' = 0, 1, 2, \dots \quad (71)$$

are

$$f_{n'}(r) = H_B \left( k - 2, cb^{-3/2}, 2n' + k, 2ab^{-1/2}, \sqrt{b}r \right) = \sum_{j=0}^{n'} C_j r^j, \quad n' = 0, 1, 2, \dots \quad (72)$$

where the coefficients  $C_j$  are easily computed by means of the three-term recurrence relations, using (67),

$$(j+1)(j+k-1)C_{j+1} + \left( \frac{(1-k)c}{2b} - a - \frac{c}{b}j \right) C_j + 2b(n'-j+1)C_{j-1} = 0, \quad C_{-1} = 0, \quad C_0 = 1, \quad (73)$$

subject to the termination condition  $C_{j+1} = 0$ . Thus, the first few polynomial solutions are given explicitly as

$$f_0(r) = 1, \quad \text{providing} \quad 2ab + (k-1)c = 0, \quad (74)$$

$$f_1(r) = 1 + \frac{2ab + (k-1)c}{2b(k-1)}r, \quad \text{providing} \quad 8b^3(1-k) + 4b^2a^2 + 4kcba + c^2(k^2-1) = 0, \quad (75)$$

$$f_2(r) = 1 + \frac{2ab + (k-1)c}{2b(k-1)}r + \frac{16b^3(1-k) + 4b^2a^2 + 4kcba + c^2(k^2-1)}{8b^2k(k-1)}r^2, \quad \text{providing} \\ 32a(1-2k)b^4 + 8(a^3 - 2c(-1+k)(3+2k))b^3 + 12a^2c(1+k)b^2 - 2ac^2(1-3k(2+k))b + c^3(k^2-1)(k+3) = 0. \quad (76)$$

For arbitrary values of the potential parameters, we may initiate the asymptotic iteration method to solve the eigenvalue problem independently of the above mentioned constraints. Although, AIM was applied previously to study this potential [2, 5], we claim that we obtain here more accurate and consistent numerical results. Using AIM with

$$\lambda_0(r) = \frac{1-k}{r} + 2br + \frac{c}{b} \quad \text{and} \quad s_0(r) = \frac{c(k-1) + 2ab}{2br} + \frac{4b^3k - c^2 - 4b^2E_{nl}^d}{4b^2}, \quad (77)$$

and computing the AIM sequences  $\lambda_n$  and  $s_n$  using (35), we evaluate, recursively, the roots of the termination condition (38), starting with the initial value  $r_0 = 3$ , similar to the technique used to report Table II. In Table III, we use AIM to verify the ‘exact’ ground state energy (74) for  $a = b = 1$ , then apply AIM to the higher excited states. In Table IV, it is clear that we have greatly improved on the earlier AIM results of Barakat [5]. These results also highlight the conclusion obtained by Amore et. al. [2] on the fast convergence of AIM for this particular problem. In Table V using the Riccati-Padé method (RPM), we report a simple comparison comparing our results with those obtained earlier by Amore et. al. [2]. An immediate reason for the improvement noted in the results of Tables IV and V a consequence of the appropriate structures of the asymptotic solutions near zero and infinity (40). This illustrates the importance of using a more adequate asymptotic solution [10] that usually yields better stability, convergence, and accuracy of AIM.

TABLE III: Eigenvalues  $E_{n0}^{d=3,4,5,6}$  for  $V(r) = 1/r + cr + r^2$  where  $c$  is determined from (74). The initial value used by AIM is  $r_0 = 3$ . The subscript  $N$  refers to the number of iteration used by AIM.

$c$	$n$	$E_{n0}^{d=3}$	$c$	$n$	$E_{n0}^{d=4}$
-1	0	2.750 000 000 000 000 000 <sub><math>N=3, exact</math></sub>	-2/3	0	3.888 888 888 888 888 888 <sub><math>N=3</math></sub> <i>Exact</i>
	1	6.105 909 691 182 920 708 <sub><math>N=64</math></sub>		1	7.485 841 099 550 171 275 <sub><math>N=59</math></sub>
	2	9.615 295 284 487 204 826 <sub><math>N=62</math></sub>		2	11.169 992 576 098 137 834 <sub><math>N=58</math></sub>
	3	13.210 469 278 706 047 371 <sub><math>N=61</math></sub>		3	14.905 199 749 925 709 834 <sub><math>N=57</math></sub>
	4	16.860 555 849 138 091 010 <sub><math>N=59</math></sub>		4	18.674 207 558 484 831 292 <sub><math>N=57</math></sub>
	5	20.549 102 541 238 464 811 <sub><math>N=57</math></sub>		5	22.467 438 445 946 572 167 <sub><math>N=55</math></sub>
	6	24.266 299 867 653 311 177 <sub><math>N=59</math></sub>		6	26.279 004 288 368 339 228 <sub><math>N=54</math></sub>
$c$	$n$	$E_{n0}^{d=5}$	$c$	$n$	$E_{n0}^{d=6}$
-1/2	0	4.937 500 000 000 000 000 <sub><math>N=3, exact</math></sub>	-2/5	0	5.960 000 000 000 000 000 <sub><math>N=3</math></sub> <i>Exact</i>
	1	8.655 823 170 124 162 086 <sub><math>N=56</math></sub>		1	9.749 149 491 375 024 656 <sub><math>N=50</math></sub>
	2	12.428 555 074 489 786 355 <sub><math>N=54</math></sub>		2	13.574 797 401 850 504 632 <sub><math>N=50</math></sub>
	3	16.234 977 694 977 922 106 <sub><math>N=52</math></sub>		3	17.424 191 007 631 759 307 <sub><math>N=49</math></sub>
	4	20.064 504 343 130 534 075 <sub><math>N=52</math></sub>		4	21.290 390 825 325 282 344 <sub><math>N=48</math></sub>
	5	23.910 981 048 253 203 499 <sub><math>N=53</math></sub>		5	25.169 188 410 054 967 854 <sub><math>N=48</math></sub>
	6	27.770 499 635 352 076 648 <sub><math>N=50</math></sub>		6	29.057 829 460 632 247 615 <sub><math>N=48</math></sub>

TABLE IV: A comparison between selected eigenenergies calculated by Barakat [5] and in the present work.

$n$	$\ell$	$a$	$c$	$b$	$E_{nl}^3$	$\epsilon_{Barakat}$
0	0	-2	0.894 42	$\sqrt{0.2}$	0.341 633 800 749 479 644 <sub><math>N=45</math></sub>	0.341 64
0	1	-2	0.447 22	$\sqrt{0.2}$	1.986 079 419 684 181 694 <sub><math>N=33</math></sub>	1.986 06
0	2	-2	0.298 14	$\sqrt{0.2}$	3.019 378 385 388 245 576 <sub><math>N=30</math></sub>	3.019 38
0	3	-2	0.223 60	$\sqrt{0.2}$	3.962 403 145 424 275 957 <sub><math>N=37</math></sub>	3.962 42
0	0	-2	8.944 28	$2\sqrt{5}$	12.416 411 447 380 603 566 <sub><math>N=83</math></sub>	6.208 20
0	1	-2	4.472 14	$2\sqrt{5}$	22.110 682 447 511 408 897 <sub><math>N=72</math></sub>	22.110 64
0	2	-2	2.981 42	$2\sqrt{5}$	31.193 837 323 750 444 679 <sub><math>N=64</math></sub>	31.193 86
0	3	-2	2.236 06	$2\sqrt{5}$	40.186 716 024 771 681 149 <sub><math>N=60</math></sub>	20.093 36

TABLE V: A comparison between eigenenergies as obtained by Amore et. al. [2] using RiccatiPadé method (RPD) and those of the present work  $E_{AIM}$ , for particular values of the parameter  $c$  in the potential  $V(r) = -2/r + cr + \sqrt{2}r^2$ .

$c$	$E_{RPM}$	$E_{AIM}$
-4	-2.343 347 169 439 4	-2.343 347 169 439 302 087 596 937 <sub><math>N=93</math></sub>
-2	-0.452 373750 381 743 8	-0.452 373 750 381 743 858 907 206 <sub><math>N=89</math></sub>
2	2.665 690 984 529 681 669 8	2.665 690 984 529 681 669 856 944 <sub><math>N=74</math></sub>
4	4.029 812 452 923 474 112 0	4.029 812 452 923 474 111 929 868 <sub><math>N=66</math></sub>

## VII. EXACT AND APPROXIMATE SOLUTIONS WITH HARD CONFINEMENT $r \leq R$ .

In this section, we turn our attention to study the  $d$ -dimensional radial Schrödinger equation

$$\left[ -\frac{d^2}{dr^2} + \frac{(k-1)(k-3)}{4r^2} + V(r) \right] u_{nl}^d(r) = E_{nl}^d u_{nl}^d(r), \quad \int_0^R |u_{nl}^d(r)|^2 dr < \infty, \quad u_{nl}^{(d)}(0) = u_{nl}^{(d)}(R) = 0, \quad (78)$$

with the potential

$$V(r) = \begin{cases} \frac{a}{r+\beta} + cr + b^2 r^2, & \text{if } 0 < r < R, \\ \infty, & \text{if } r \geq R, \end{cases} \quad (79)$$

where  $u_{nl}^d(0) = u_{nl}^d(R) = 0$ . We employ the following ansatz for the wave function

$$u_{nl}^d(r) = r^{(k-1)/2} (R-r) \exp\left(-\frac{c}{2b}r - \frac{b}{2}r^2\right) f_n(r), \quad k = d + 2l, \quad (80)$$

where the  $(R-r)$  factor is inserted to ensure the vanishing of the radial wave function  $u_{nl}^d(r)$  at the boundary  $r = R$ . On substituting (80) into (78), we obtain the following second-order differential equation for the functions  $f_n(r)$ ,

$$\begin{aligned} & \left( -4b^2 r^3 + 4b^2(R-\beta)r^2 + 4b^2\beta Rr \right) f_n''(r) + \left( 8b^3 r^4 + 4b(c + 2b^2(\beta - R))r^3 - 4b(b - \beta c + bk + 2b^2\beta R + cR)r^2 \right. \\ & \quad \left. - 4b(\beta cR + b(\beta + \beta k + R - kR))r + 4b^2\beta(k-1)R \right) f_n'(r) \\ & + \left[ (4b^3(2+k) - c^2 - 4b^2E)r^3 + (4ab^2 + 2bc(1+k) + (c^2 + 4b^2E)(R-\beta) + 4b^3(\beta(k+2) - kR))r^2 \right. \\ & \quad \left. + (2b(\beta c(1+k) - 2b(k-1)) + (\beta(c^2 + 4b^2E) - 4ab^2 - 2bc(k-1) - 4b^3\beta k)R)r - 2b\beta(k-1)(2b + cR) \right] f_n(r) = 0. \end{aligned} \quad (81)$$

This differential equation goes beyond the equation discussed in section III, so we introduce another more general class of differential equation that allows us to analyze the polynomial solutions of (81).

**Theorem VII.1.** *The second-order linear differential equation*

$$(a_{5,0}r^5 + a_{5,1}r^4 + a_{5,2}r^3 + a_{5,3}r^2 + a_{5,4}r + a_{5,5})f''(r) + (a_{4,0}r^4 + a_{4,1}r^3 + a_{4,2}r^2 + a_{4,3}r + a_{4,4})f'(r) - (\tau_{3,0}r^3 + \tau_{3,1}r^2 + \tau_{3,2}r + \tau_{3,3})f(r) = 0 \quad (82)$$

has a polynomial solution  $y(r) = \sum_{j=0}^n c_j r^j$ , if

$$\tau_{3,0} = n(n-1)a_{5,0} + na_{4,0}, \quad n = 0, 1, 2, \dots, \quad (83)$$

provided  $a_{5,0}^2 + a_{4,0}^2 + \tau_{3,0}^2 \neq 0$ . The polynomial coefficients  $c_n$  then satisfy the following six-term recurrence relation

$$\begin{aligned} & ((j-3)(j-4)a_{5,0} + (j-3)a_{4,0} - \tau_{3,0})c_{j-3} + ((j-2)(j-3)a_{5,1} + (j-2)a_{4,1} - \tau_{3,1})c_{j-2} \\ & + ((j-1)(j-2)a_{5,2} + (j-1)a_{4,2} - \tau_{3,2})c_{j-1} + (j(j-1)a_{5,3} + ja_{4,3} - \tau_{3,3})c_j \\ & + (j(j+1)a_{5,4} + (j+1)a_{4,4})c_{j+1} + (j+1)(j+2)a_{5,5}c_{j+2} = 0 \end{aligned} \quad (84)$$

with  $c_{-3} = c_{-2} = c_{-1} = 0$ . In particular, for the zero-degree polynomials  $f_0(r) = 1$  where  $c_0 = 1$  and  $c_n = 0$ ,  $n \geq 1$ , we must have  $\tau_{3,0} = 0$  along with

$$\tau_{3,1} = 0, \quad \tau_{3,2} = 0, \quad \tau_{3,3} = 0. \quad (85)$$

For the first-degree polynomial solution

$$f_1(r) = 1 + \frac{\tau_{3,3}}{a_{4,4}}r,$$

where  $c_0 = 1$ ,  $c_1 = \tau_{3,3}/a_{4,4}$ , and  $c_n = 0$ ,  $n \geq 2$ , we must have  $\tau_{3,0} = a_{4,0}$  along with the vanishing of the three  $2 \times 2$ -determinants, simultaneously,

$$\begin{vmatrix} -\tau_{3,3} & a_{4,4} \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} \end{vmatrix} = 0, \quad \begin{vmatrix} -\tau_{3,3} & a_{4,4} \\ -\tau_{3,1} & a_{4,2} - \tau_{3,2} \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} -\tau_{3,3} & a_{4,4} \\ -a_{4,0} & a_{4,1} - \tau_{3,1} \end{vmatrix} = 0. \quad (86)$$

For the second-degree polynomial solution,

$$f_2(r) = 1 + \frac{(a_{4,4} + a_{5,4})\tau_{3,3} - a_{5,5}\tau_{3,2}}{a_{4,4}(a_{4,4} + a_{5,4}) + a_{5,5}(\tau_{3,3} - a_{4,3})} r + \frac{a_{4,4}\tau_{3,2} + \tau_{3,3}(\tau_{3,3} - a_{4,3})}{2(a_{4,4}(a_{4,4} + a_{5,4}) + a_{5,5}(\tau_{3,3} - a_{4,3}))} r^2$$

where  $c_n = 0$  for  $n \geq 3$ , we must have  $\tau_{3,0} = 2a_{5,0} + 2a_{4,0}$  along with the vanishing of the three  $3 \times 3$ -determinants, simultaneously,

$$\begin{vmatrix} -\tau_{3,3} & a_{4,4} & 2a_{5,5} \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} & 2a_{5,4} + 2a_{4,4} \\ -\tau_{3,1} & a_{4,2} - \tau_{3,2} & 2a_{5,3} + 2a_{4,3} - \tau_{3,3} \end{vmatrix} = 0, \quad \begin{vmatrix} -\tau_{3,3} & a_{4,4} & 2a_{5,5} \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} & 2a_{5,4} + 2a_{4,4} \\ -2a_{5,0} - 2a_{4,0} & a_{4,1} - \tau_{3,1} & 2a_{5,2} + 2a_{4,2} - \tau_{3,2} \end{vmatrix} = 0, \quad (87)$$

and

$$\begin{vmatrix} -\tau_{3,3} & a_{4,4} & 2a_{5,5} \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} & 2a_{5,4} + 2a_{4,4} \\ 0 & -2a_{5,0} - a_{4,0} & 2a_{5,1} + 2a_{4,1} - \tau_{3,1} \end{vmatrix} = 0, \quad (88)$$

For third-degree polynomial solution,

$$\begin{aligned} f_3(r) = 1 + & \frac{2a_{5,5}^2\tau_{3,1} + (a_{4,4} + a_{5,4})(a_{4,4} + 2a_{5,4})\tau_{3,3} + a_{5,5}(\tau_{3,3}^2 - 2(a_{4,3} + a_{5,3})\tau_{3,3} - (a_{4,4} + 2a_{5,4})\tau_{3,2})}{a_{4,4}^3 + 3a_{4,4}^2a_{5,4} + 2a_{5,5}(-a_{4,3}a_{5,4} + a_{4,2}a_{5,5} - a_{5,5}\tau_{3,2} + a_{5,4}\tau_{3,3}) + a_{4,4}(2a_{5,4}^2 + a_{5,5}(2\tau_{3,3} - 3a_{4,3} - 2a_{5,3}))} r \\ & + \frac{a_{4,4}^2\tau_{3,2} + 2\tau_{3,3}(-a_{4,3}a_{5,4} + a_{4,2}a_{5,5} - a_{5,5}\tau_{3,2} + a_{5,4}\tau_{3,3}) + a_{4,4}(-2a_{5,5}\tau_{3,1} + 2a_{5,4}\tau_{3,2} - a_{4,3}\tau_{3,3} + \tau_{3,3}^2)}{2(a_{4,4}^3 + 3a_{4,4}^2a_{5,4} + 2a_{5,5}(-a_{4,3}a_{5,4} + a_{4,2}a_{5,5} - a_{5,5}\tau_{3,2} + a_{5,4}\tau_{3,3}) + a_{4,4}(2a_{5,4}^2 + a_{5,5}(2\tau_{3,3} - 3a_{4,3} - 2a_{5,3})))} r^2 \\ & + \frac{\mu}{6(a_{4,4}^3 + 3a_{4,4}^2a_{5,4} + 2a_{5,5}(-a_{4,3}a_{5,4} + a_{4,2}a_{5,5} - a_{5,5}\tau_{3,2} + a_{5,4}\tau_{3,3}) + a_{4,4}(2a_{5,4}^2 + a_{5,5}(-3a_{4,3} - 2a_{5,3} + 2\tau_{3,3})))} r^3, \end{aligned}$$

where

$$\begin{aligned} \mu = & 2a_{4,4}^2\tau_{3,1} + 2a_{4,4}a_{5,4}\tau_{3,1} - 2a_{4,4}(a_{4,3} + a_{5,3})\tau_{3,2} - 2a_{5,5}(a_{4,3}\tau_{3,1} + \tau_{3,2}(-a_{4,2} + \tau_{3,2})) \\ & + (2a_{4,3}(a_{4,3} + a_{5,3}) - 2a_{4,2}(a_{4,4} + a_{5,4}) + 2a_{5,5}\tau_{3,1} + 3a_{4,4}\tau_{3,2} + 2a_{5,4}\tau_{3,2})\tau_{3,3} - (3a_{4,3} + 2a_{5,3})\tau_{3,3}^2 + \tau_{3,3}^3. \end{aligned} \quad (89)$$

where  $c_n = 0$  for  $n \geq 4$ , we must have  $\tau_{3,0} = 6a_{5,0} + 3a_{4,0}$  along with the vanishing of the three  $4 \times 4$ -determinants, simultaneously,

$$\begin{vmatrix} -\tau_{3,3} & a_{4,4} & 2a_{5,5} & 0 \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} & 2a_{5,4} + 2a_{4,4} & 6a_{5,5} \\ -\tau_{3,1} & a_{4,2} - \tau_{3,2} & 2a_{4,3} + 2a_{5,3} - \tau_{3,3} & 3a_{4,4} + 6a_{5,4} \\ -6a_{5,0} - 3a_{4,0} & a_{4,1} - \tau_{3,1} & 2a_{4,2} + 2a_{5,2} - \tau_{3,2} & 3a_{4,3} + 6a_{5,3} - \tau_{3,3} \end{vmatrix} = 0, \quad (90)$$

$$\begin{vmatrix} -\tau_{3,3} & a_{4,4} & 2a_{5,5} & 0 \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} & 2a_{5,4} + 2a_{4,4} & 6a_{5,5} \\ -\tau_{3,1} & a_{4,2} - \tau_{3,2} & 2a_{4,3} + 2a_{5,3} - \tau_{3,3} & 3a_{4,4} + 6a_{5,4} \\ 0 & -6a_{5,0} - 2a_{4,0} & 2a_{4,1} + 2a_{5,1} - \tau_{3,1} & 3a_{4,2} + 6a_{5,2} - \tau_{3,2} \end{vmatrix} = 0, \quad (91)$$

and

$$\begin{vmatrix} -\tau_{3,3} & a_{4,4} & 2a_{5,5} & 0 \\ -\tau_{3,2} & a_{4,3} - \tau_{3,3} & 2a_{5,4} + 2a_{4,4} & 6a_{5,5} \\ -\tau_{3,1} & a_{4,2} - \tau_{3,2} & 2a_{4,3} + 2a_{5,3} - \tau_{3,3} & 3a_{4,4} + 6a_{5,4} \\ 0 & 0 & -a_{4,0} - 4a_{5,0} & 3a_{4,1} + 6a_{5,1} - \tau_{3,1} \end{vmatrix} = 0, \quad (91)$$

and so on, for higher-order polynomial solutions. The vanishing of these determinants can be regarded as the conditions under which the coefficients  $\tau_{3,1}$ ,  $\tau_{3,2}$  and  $\tau_{3,3}$  of Eq.(82) are determined in terms of the other coefficients.

*Proof.* The proof of this theorem is rather lengthy: it employs the asymptotic iteration method in a similar way to the approach used by Saad *et al* in (2014) ([31], Appendix A).  $\square$

We shall first verify the conclusions of this theorem regarding equation (81) by using the asymptotic iteration method followed by an analysis of the solutions for arbitrary parameters. To this end, we employ AIM for (81) using

$$\lambda_0 = \frac{c}{b} + \frac{1-k}{r} + 2br - \frac{2}{r-R}, \quad s_0 = b(2+k) - \frac{c^2}{4b^2} - E + \frac{a}{\beta+r} + \frac{(k-1)(2b+cR)}{2bRr} + \frac{b(1-k)+R(c+2b^2R)}{bR(r-R)}, \quad (92)$$

and by means of

$$\begin{aligned} a_{5,0} &= a_{5,1} = a_{5,5} = 0, & a_{5,2} &= -4b^2, & a_{5,3} &= 4b^2(R-\beta), & a_{5,2} &= 4b^2\beta R \\ a_{4,0} &= 8b^3, & a_{4,1} &= 4b(c+2b^2(\beta-R)), & a_{4,2} &= -4b(b-\beta c+bk+2b^2\beta R+cR) \\ a_{4,3} &= -4b(\beta cR+b(\beta+\beta k+R-kR)), & a_{4,4} &= 4b^2\beta(k-1)R \\ \tau_{3,0} &= -(4b^3(2+k)-c^2-4b^2E), & \tau_{3,1} &= -(4ab^2+2bc(1+k)+(c^2+4b^2E)(R-\beta)+4b^3(\beta(k+2)-kR)), \\ \tau_{3,2} &= -(2b(\beta c(1+k)-2b(k-1))+(\beta(c^2+4b^2E)-4ab^2-2bc(k-1)-4b^3\beta k)R), & \tau_{3,3} &= 2b\beta(k-1)(2b+cR), \end{aligned} \quad (93)$$

the necessary condition for the existence of polynomial solutions  $f_n(r) = \sum_{k=0}^n c_k r^k$  of Eq. (81) becomes

$$E_{n\ell}^d = b(2n' + k + 2) - \frac{c^2}{4b^2}, \quad k = d + 2l, \quad (94)$$

where  $n'$  refers to the degree of the polynomial solution of equation (81) and is not necessarily equal to the number of nodes  $n$  of the wave function. It is clear from (93), there is no zero-degree polynomial solution available. For the first-degree polynomial solution, we have

$$E_{n\ell}^d = b(4+k) - \frac{c^2}{4b^2}, \quad f_1(r) = 1 + \left(\frac{c}{2b} + \frac{1}{R}\right)r \quad (95)$$

providing

$$\begin{aligned} 4ab^2 + 2bc(3+k) + (2abc + c^2(3+k))R + 4b^2cR^2 &= 0 \\ 2b(\beta c(3+k) - 4bk) + c(\beta c(3+k) - 4bk)R + (16b^3 - 2abc + 4b^2\beta c - c^2(1+k))R^2 &= 0 \\ 8b^2\beta k + 4b\beta ckR + (4ab^2 + \beta(c^2(1+k) - 16b^3))R^2 &= 0. \end{aligned} \quad (96)$$

In Table VII, we report the exact eigenvalues  $E_{00}^3 = 7b - c^2/(4b^2)$  using the roots of the equations given by (96) and the results obtained by AIM initiated with  $r_0 = R/2$  for different values of  $R$  and  $\beta$ , where we have fixed  $k = 3$ . For arbitrary values of the potential parameters, we can employ AIM initiated with (92) to obtain accurate eigenvalues as the roots of the termination condition 38. Some of these results are reported in Table VII. There is an interesting additional application of AIM for these confining potentials: it is possible to use the termination condition to find the proper radius of confinement  $R$  for a particular energy; in other words, we may regard the termination condition as function of  $(r, R)$  given a particular energy  $E$ . Consider for example  $\beta = a = b = -c = 1, k = 3$ , and  $E = 9$ , what is the radius of confinement for this particular case? The direct application of AIM implies that  $R = 1.074\,414\,209\,270\,221\,205$  while for  $E = 10$ , the proper radius of confinement  $R = 1.016\,954\,256\,339\,063\,400$ . The method can be easily generalized for arbitrary values of the parameters.

## VIII. CONCLUSION

In this work exact and approximate solutions of Schrödinger's equation with softcore Coulomb potentials under hard and soft confinement were found. These problems generate an interesting class of differential equation that goes beyond the classical problems which have solutions of hypergeometric type. In this paper the problems were analyzed as special cases of a very general scheme for the study of linear second-order differential equations with polynomial coefficients that admit polynomial solutions. Necessary and sufficient conditions are derived for the existence of such solutions. The methods presented in this work allow us to obtain compact algebraic expressions for the exact analytical solutions. These are then verified by the asymptotic iteration method. In cases where the parametric conditions for exact polynomial solutions are not met, the asymptotic iteration method is employed directly to find highly accurate

TABLE VI: A comparison between selected eigenenergies calculated using AIM with the exact values obtained as the roots of equation (96).

$a$	$b$	$c$	$\beta$	$R$	$E_{AIM}$	$E_{exact}$
20/3	11/6	-11/6	2	1	12.583 333 333 333 333 333 <sub>N=3</sub>	151/12
28/15	13/30	-13/90	3	2	3.005 555 555 555 555 556 <sub>N=3</sub>	541/180
55/63	47/252	-47/1512	4	3	1.298 611 111 111 111 111 <sub>N=3</sub>	187/144
143/90	71/360	-71/1350	5	3	1.362 777 777 777 777 778 <sub>N=3</sub>	2453/1800
91/180	37/360	-37/3600	5	4	0.716 944 444 444 444 444 <sub>N=3</sub>	2581/3600
14/15	13/120	-13/720	6	4	0.751 388 888 888 888 889 <sub>N=3</sub>	541/720

TABLE VII: Eigenvalues  $E_{n\ell}^{d=3}$  for  $V(r) = 1/(r+1) - r + r^2$  for different radius of confinement  $R$ . The initial value employed by AIM is  $r_0 = R/2$ . The subscript  $N$  refers to the number of iterations used by AIM.

$R$	$n$	$E_{n0}^{d=3}$	$R$	$n$	$E_{n0}^{d=3}$
1	0	10.328 716 871 106 505 751 <sub>N=36</sub>	2	0	3.105 413 452 488 593 322 <sub>N=56</sub>
	1	39.987 716 212 123 541 087 <sub>N=37</sub>		1	10.692 851 715 920 035 023 <sub>N=55</sub>
	2	89.345 269 629 504 444 833 <sub>N=36</sub>		2	23.063 954 484 826 017 705 <sub>N=57</sub>
	3	158.435 845 294 778 224 568 <sub>N=41</sub>		3	40.347 069 688 147 624 366 <sub>N=55</sub>
$R$	$\ell$	$E_{0\ell}^{d=3}$	$R$	$\ell$	$E_{0\ell}^{d=3}$
1	0	10.328 716 871 106 505 751 <sub>N=36</sub>	2	0	3.105 413 452 488 593 322 <sub>N=56</sub>
	1	20.608 236 713 301 997 322 <sub>N=36</sub>		1	5.819 309 536 633 945 722 <sub>N=55</sub>
	2	33.620 107 194 959 911 851 <sub>N=36</sub>		2	9.196 161 676 541 214 605 <sub>N=56</sub>
	3	49.228 314 838 693 690 037 <sub>N=36</sub>		3	21.521 551 806 858 223 355 <sub>N=59</sub>

numerical solutions. In this work, the asymptotic iteration method served two purposes. The first was to confirm the validity of the sufficient conditions obtained analytically. The second is to provide approximate solutions to the eigenvalue problems, whether potential parameters are specially restricted or freely chosen. For both purposes, the method proves to be extremely effective and provides very accurate results. It is also clear from the present work that the method and the analytic expressions obtained for the different classes of the differential equations can be easily adapted to study other eigenproblems appearing in theoretical physics.

## IX. ACKNOWLEDGMENTS

Partial financial support of this work under Grant Nos. GP3438 and GP249507 from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by us (respectively RLH and NS).

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